

TINGLEY'S PROBLEM THROUGH THE FACIAL STRUCTURE OF AN ATOMIC JBW*-TRIPLE

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ABSTRACT. We prove that every surjective isometry between the unit spheres of two atomic JBW*-triples E and B admits a unit extension to a surjective real linear isometry from E into B . This result constitutes a new positive answer to Tingley's problem in the Jordan setting.

1. INTRODUCTION

In a recent contribution we establish that every surjective isometry between the unit spheres of two $B(H)$ -spaces extends uniquely to a surjective complex linear or conjugate linear surjective isometry between the corresponding spaces (see [15]). This result constitutes a positive answer to Tingley's isometric extension problem [31] in the setting of $B(H)$ -spaces and atomic von Neumann algebras. Solutions to Tingley's problem for compact operators, compact C^* -algebras and weakly compact JB*-triples have been previously obtained in [26, 14]. For additional information on the historic background and the state of the art of Tingley's problem the reader is referred to the introduction of [15] and to the monograph [32].

Problems in C^* -algebras, von Neumann algebras and operator algebras are often considered in the context of Banach Jordan algebras and Jordan triple systems. Such studies widen the scope and often introduce new ideas and techniques not present in the associative case. The class of JB*-triples have a rich interaction with Banach Space Theory. The spaces in this class enjoy a unique geometry which makes more interesting the study of certain geometric problems in a wider setting. This paper is devoted to extend the recent results in [15] to the context of atomic JBW*-triples (i.e. ℓ_∞ -sums of Cartan factors).

We recall that a *JB*-triple* is a complex Banach space E which can be equipped with a continuous triple product $\{., ., .\} : E \times E \times E \rightarrow E$, which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies the following axioms

- (a) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, where $L(a, b)$ is the operator on E given by $L(a, b)x = \{a, b, x\}$;
- (b) $L(a, a)$ is an hermitian operator with non-negative spectrum;
- (c) $\|L(a, a)\| = \|a\|^2$.

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Examples of JB*-triples include C*-algebras with respect to the triple product defined by product

$$(1) \quad \{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x),$$

and JB*-algebras under the triple product $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$. The so-called *ternary rings of operators* (TRO's) studied, for example, in [25] are also examples of JB*-triples.

A subtriple \mathcal{I} of a JB*-triple E is said to be an *ideal* or a *triple ideal* of E if $\{E, E, \mathcal{I}\} + \{E, \mathcal{I}, E\} \subseteq \mathcal{I}$.

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [2]). It is known that the second dual of a JB*-triple is a JBW*-triple (compare [8]). An extension of Sakai's theorem assures that the triple product of every JBW*-triple is separately weak*-continuous (c.f. [2] or [19]).

Another illustrative examples of JBW*-triples are given by the so-called Cartan factors. A complex Banach space is a *Cartan factor of type 1* if it coincides with the complex Banach space $L(H, K)$, of all bounded linear operators between two complex Hilbert spaces, H and K , whose triple product is given by (1).

Given a conjugation, j , on a complex Hilbert space, H , we can define a linear involution on $L(H)$ defined by $x \mapsto x^t := jx^*j$. A *type 2 Cartan factor* is a subtriple of $L(H)$ formed by the skew-symmetric operators for the involution t ; similarly, a *type 3 Cartan factor* is formed by the t -symmetric operators. A Banach space X is called a *Cartan factor of type 4* or *spin* if X admits a complete inner product $(\cdot|\cdot)$ and a conjugation $x \mapsto \overline{x}$, for which the norm of X is given by

$$\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|\overline{x})|^2}.$$

Cartan factors of types 5 and 6 (also called *exceptional* Cartan factors) are all finite dimensional. An atomic JBW*-triple is a JBW*-triple which can be represented as an ℓ_∞ -sum of Cartan factors. We refer to [7] for additional details.

Let E and B be atomic JBW*-triples. In our main result we prove that every surjective isometry $f : S(E) \rightarrow S(B)$ admits a unique extension to a surjective real linear isometry $T : E \rightarrow B$ (see Theorem 2.9). This theorem extends the main conclusion in [15] to the setting of atomic JBW*-triples. In [15], we strived for arguments essentially based on standard techniques of C*-algebras, Geometry and Functional Analysis. The proofs here are extended to the setting of JBW*-triples with new and independent techniques which could be also applied to C*-algebras.

As in recent contributions studying Tingley's problem on C*-algebras and von Neumann algebras, our arguments are based on an useful result due to L. Cheng, Y. Dong and R. Tanaka, which asserts that every surjective isometry between the unit spheres of two Banach spaces X and Y maps maximal proper (norm closed) face of the unit ball of X to maximal proper (norm closed) face of the unit ball of Y (see [6, Lemma 5.1], [27, Lemma 3.5] and [28, Lemma 3.3]).

Throughout the paper, given a Banach space X the symbols \mathcal{B}_X and $S(X)$ will stand for the closed unit ball and the unite sphere of X , respectively.

By a result of C.M. Edwards, C. Hoskin and the authors of this note (see [9]) we know that for each non-empty norm closed face F of the unit ball \mathcal{B}_E in a JB*-triple

E there exists a unique compact tripotent u in E^{**} such that

$$F = F_u = (u + E_0^{**}(u)) \cap \mathcal{B}_E,$$

where $E_0^{**}(u)$ is the Peirce-zero space associated with u in E^{**} .

An appropriate application of Kadison's transitivity theorem for JB*-triples ([4, Theorems 3.3 and 3.4]) proves that every maximal proper norm closed face of \mathcal{B}_E is of the form

$$(2) \quad F_e = (e + E_0^{**}(e)) \cap \mathcal{B}_E,$$

for a unique minimal tripotent e in E^{**} . However this minimal tripotent e need not be in E .

We recall that every atomic JBW*-triple coincides with the weak*-closure of the linear span of all its minimal tripotents (see [17]).

If B is another atomic JBW*-triple and $f : S(E) \rightarrow S(B)$ is a surjective isometry, for each minimal tripotent $e \in E$, there exists a minimal tripotent u in B^{**} satisfying

$$f(F_e) = f((e + E_0(e)) \cap \mathcal{B}_E) = (u + B_0^{**}(u)) \cap \mathcal{B}_B.$$

However, as in the case of Tingley's theorem for surjective isometries between the unit spheres of $B(H)$ -spaces (see [15]), when dealing with maximal proper faces in \mathcal{B}_B , minimal tripotents in B^{**} need not be in B . To avoid this difficulty, in this paper we shall prove that, under the above conditions, the minimal tripotent u belongs to B (see Theorem 2.7). The proofs in this note are based on geometric arguments combined with Functional Analysis techniques.

2. TINGLEY'S PROBLEM FOR ATOMIC JBW*-TRIPLES

An element u in a JB*-triple E is called tripotent if $\{u, u, u\} = u$. Each tripotent u in E induces a *Peirce decomposition* of E in the form

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u),$$

where for each $i \in \{0, 1, 2\}$ the space $E_i(u)$ is precisely the $\frac{i}{2}$ eigenspace of $L(u, u)$. Peirce arithmetic assures that $\{E_i(u), E_j(u), E_k(u)\}$ is contained in $E_{i-j+k}(u)$ if $i - j + k \in \{0, 1, 2\}$, and is zero otherwise. In addition,

$$\{E_2(u), E_0(u), E\} = \{E_0(u), E_2(u), E\} = 0.$$

The corresponding *Peirce projections*, $P_i(u) : E \rightarrow E_i(u)$, ($i = 0, 1, 2$) are contractive and satisfy

$$P_2(u) = L(u, u)(2L(u, u) - I), \quad P_1(u) = 4L(u, u)(I - L(u, u)),$$

and $P_0(u) = (I - L(u, u))(I - 2L(u, u))$ (compare [16]).

A non-zero tripotent e in a JB*-triple E is called *minimal* (respectively, *complete* or *maximal*) if $E_2(e) = \mathbb{C}e$ (respectively, $E_0(e) = 0$).

Let x be an element in a JB*-triple E . We will denote by E_x the JB*-subtriple of E generated by x , that is, the closed subspace generated by all odd powers of the form $x^{[1]} := x$, $x^{[3]} := \{x, x, x\}$, and $x^{[2n+1]} := \{x, x, x^{[2n-1]}\}$, ($n \in \mathbb{N}$). It is known that E_x is JB*-triple isomorphic (and hence isometric) to a commutative C*-algebra in which x is a positive generator (cf. [21, Corollary 1.15]). Actually, there exist a (unique) subset $\text{Sp}(x) \subset [0, \|x\|]$ (called the *triple spectrum* of x)

such that $\|x\| \in \text{Sp}(x) \cup \{0\}$ and the latter is compact, and a triple isomorphism $\Psi : E_x \rightarrow C_0(\text{Sp}(x))$ mapping x into the function $t \mapsto t$ (compare [22]).

Suppose x is a norm-one element. The sequence $(x^{[2n-1]})$ converges in the weak*-topology of E^{**} to a tripotent (called the *support tripotent* of x) $u(x)$ in E^{**} (see [10, Lemma 3.3] or [9, page 130]).

We recall at this stage a result taken from [14].

Proposition 2.1. [14, Proposition 2.2] *Let e and x be norm-one elements in a JB^* -triple E . Suppose that e is a minimal tripotent and $\|e - x\| = 2$. Then the identity $x = -e + P_0(e)(x)$ holds.* \square

Following the standard notation, for each ultrafilter \mathcal{U} on an index set I , and each family $(X_i)_{i \in I}$ of Banach spaces, we denote by $(X_i)_{\mathcal{U}}$ the corresponding ultraproduct of the X_i , and if $X_i = X$ for all i , we write $(X)_{\mathcal{U}}$ for the ultrapower of X . As usually, elements in $(X_i)_{\mathcal{U}}$ will be denoted in the form $\tilde{x} = [x_i]_{\mathcal{U}}$, where (x_i) is called a representing family or a representative of \tilde{x} , and $\|\tilde{x}\| = \lim_{\mathcal{U}} \|x_i\|$ independently of the representative. The basic facts and definitions concerning ultraproducts can be found in [18].

In our concrete setting, we can reduce our attention to a family $(E_i)_{i \in I}$ of JB^* -triples. It is known that JB^* -triples are stable under ℓ_{∞} -sums (see [21, p. 523]), thus the Banach space $\ell_{\infty}(E_i)$ is a JB^* -triple with pointwise operations. The closed subtriple $c_0(E_i)$ is a triple ideal of $\ell_{\infty}(E_i)$, and hence $(E_i)_{\mathcal{U}} = \ell_{\infty}(E_i)/c_0(E_i)$ is a JB^* -triple (see [21]).

Our next goal is a quantitative version of the previous Proposition 2.1.

Proposition 2.2. *Let e be a minimal tripotent in a JB^* -triple E . Then for each $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following property: for each $x \in S(E)$ with $\|e - x\| > 2 - \delta$ we have $\|P_2(e)(e - x)\| > 2 - \varepsilon$.*

Proof. Arguing by reduction to the absurd, we assume the existence of $\varepsilon > 0$ such that for each natural n there exists $x_n \in S(E)$ with $\|e - x_n\| > 2 - \frac{1}{n}$ and $\|P_2(e)(e - x_n)\| \leq 2 - \varepsilon$, for every natural n .

Let \mathcal{U} be a free ultrafilter over \mathbb{N} . We consider the JB^* -triple $E_{\mathcal{U}}$. Clearly the element $[e]_{\mathcal{U}}$ is a tripotent in $E_{\mathcal{U}}$. We claim that $[e]_{\mathcal{U}}$ is a minimal tripotent. Indeed, suppose $\tilde{x} = [x_n]_{\mathcal{U}} \in (E_{\mathcal{U}})_2([e]_{\mathcal{U}})$, then

$$[x_n]_{\mathcal{U}} = P_2([e]_{\mathcal{U}})[x_n]_{\mathcal{U}} = [P_2(e)(x_n)]_{\mathcal{U}} = [\lambda_n e]_{\mathcal{U}},$$

where (λ_n) is a bounded family in \mathbb{C} . By compactness arguments the limit $\lim_{\mathcal{U}} (\lambda_n) = \lambda_0$ exists in \mathbb{C} . By the triangular inequality we have

$$\|[x_n]_{\mathcal{U}} - \lambda_0 [e]_{\mathcal{U}}\| \leq \|[x_n]_{\mathcal{U}} - [\lambda_n e]_{\mathcal{U}}\| + \|[\lambda_n e]_{\mathcal{U}} - \lambda_0 [e]_{\mathcal{U}}\| = \lim_{\mathcal{U}} |\lambda_n - \lambda_0|,$$

and hence $[x_n]_{\mathcal{U}} = \lambda_0 [e]_{\mathcal{U}} \in \mathbb{C}[e]_{\mathcal{U}}$.

Finally, since $\|[x_n]_{\mathcal{U}}\| = 1$, and $2 \geq \|[e]_{\mathcal{U}} - [x_n]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|e - x_n\| \geq \lim_{\mathcal{U}} 2 - \frac{1}{n} = 2$. Applying Proposition 2.1 we get

$$[x_n]_{\mathcal{U}} = [-e]_{\mathcal{U}} + P_0([e]_{\mathcal{U}})[x_n]_{\mathcal{U}} = [-e + P_0(e)(x_n)]_{\mathcal{U}},$$

and thus

$$[P_2(e)(e - x_n)]_{\mathcal{U}} = P_2([e]_{\mathcal{U}})([e]_{\mathcal{U}} - [x_n]_{\mathcal{U}}) = 2[e]_{\mathcal{U}},$$

and

$$\lim_{\mathcal{U}} \|P_2(e)(e - x_n)\| = \lim_{\mathcal{U}} \|2e\| = 2,$$

which contradicts $\|P_2(e)(e - x_n)\| \leq 2 - \varepsilon$, for every natural n . \square

A common tool applied in all recent studies on Tingley's problem for non-commutative structures is based on an appropriate description of the facial structures of the closed unit balls of the involved spaces. The justification is essentially due to the following result established by L. Cheng, Y. Dong and R. Tanaka.

Proposition 2.3. ([6, Lemma 5.1], [27, Lemma 3.5] and [28, Lemma 3.3]) *Let X, Y be Banach spaces, and let $T : S(X) \rightarrow S(Y)$ be a surjective isometry. Then C is a maximal convex subset of $S(X)$ if and only if $T(C)$ is that of $S(Y)$. Then C is a maximal proper (norm closed) face of \mathcal{B}_X if and only if $f(C)$ is a maximal proper (norm closed) face of \mathcal{B}_Y .* \square

Accordingly to the notation in [11] and [13], a tripotent e in the second dual, E^{**} , of a JB*-triple E is said to be *compact- G_δ* if there exists a norm one element a in E such that e is the support tripotent of a . A tripotent e in E^{**} is called *compact* if $e = 0$ or it is the infimum of a decreasing net of compact- G_δ tripotents in E^{**} .

The norm closed faces of the closed unit ball of a C*-algebra were determined by C.A. Akemann and G.K. Pedersen in [1]. Their characterization played a decisive role in the arguments presented in [28, 29, 30, 26] and [15]. The result of Akemann and Pedersen was extended to the strictly wider setting of JB*-triples by C.M. Edwards, C. Hoskin and the authors of this note in [9]. For later purposes we recall a theorem borrowed from the just quoted paper.

Theorem 2.4. [9] *Let E be a JB*-triple, and let F be a non-empty norm closed face of the unit ball \mathcal{B}_E in E . Then, there exists a unique compact tripotent u in E^{**} such that*

$$F = F_u = (u + E_0^{**}(u)) \cap \mathcal{B}_E,$$

where $E_0^{**}(u)$ is the Peirce-zero space associated with u in E^{**} . Furthermore, the mapping $u \mapsto F_u$ is an anti-order isomorphism from the lattice $\tilde{\mathcal{U}}_c(E^{**})$ of all compact tripotents in E^{**} onto the complete lattice of norm closed faces of \mathcal{B}_E . \square

Let e be a tripotent in a JB*-triple E . We shall say that e is a *finite-rank tripotent* if e can be written as a finite sum of mutually orthogonal minimal tripotents in E . An appropriate extension of Kadison's transitivity theorem for JB*-triples, established in [4, Theorems 3.3 and 3.4], proves that each finite-rank tripotent e in the bidual E^{**} of E is compact, and it is further known that

$$(3) \quad P_2(e)(E^{**}) = P_2(e)(E) = \mathbb{C}e, \text{ and } P_1(e)(E^{**}) = P_1(e)(E),$$

where $P_j(e)$ stands for the j -th Peirce projection associated with e in E^{**} . Accordingly to these comments and the previous Theorem 2.4, every maximal proper norm closed face of \mathcal{B}_E is of the form

$$(4) \quad F_e = (e + E_0^{**}(e)) \cap \mathcal{B}_E,$$

for a unique minimal tripotent e in E^{**} . However this minimal tripotent e need not be in E .

We recall that elements a, b in a JB*-triple E are said to be *orthogonal* (written $a \perp b$) if $L(a, b) = 0$ (see [5, Lemma 1] for additional details). It follows from Peirce arithmetic that, for each tripotent $e \in E$, $E_2(e) \perp E_0(e)$. The relation "being orthogonal" can be applied to define a partial order in the set of tripotents in E

given by $u \leq e$ if $e - u$ is a tripotent orthogonal to e (see, for example, [16] or [19]). It is known that in a JBW*-triple M a tripotent $e \in M$ is minimal if and only if it is minimal for the order \leq .

We are now in position to extend [15, Theorem 2.3].

Theorem 2.5. *Let E and B be JB*-algebras, and suppose that $f : S(E) \rightarrow S(B)$ is a surjective isometry. Let e be a minimal tripotent in E . Then 1 is isolated in the triple spectrum of $f(e)$.*

Proof. Arguing by contradiction, we assume that 1 is not isolated in $\text{Sp}(f(e))$. We shall identify $B_{f(e)}$ with $C_0(\text{Sp}(f(e)))$.

By Proposition 2.3, Theorem 2.4, and (4), there exists a minimal tripotent $u \in B^{**}$ such that

$$(5) \quad f(F_e) = f((e + E_0(e)) \cap \mathcal{B}_E) = f((e + E_0^{**}(e)) \cap \mathcal{B}_E) = F_u = (u + B_0^{**}(e)) \cap \mathcal{B}_B.$$

For each natural n , we define \hat{x}_n, \hat{y}_n the elements in $B_{f(e)}$ given by:

$$\hat{x}_n(t) := \begin{cases} \frac{t}{t_0}, & \text{if } 0 \leq t \leq 1 - \frac{1}{n} \\ \text{affine}, & \text{if } 1 - \frac{1}{n} \leq t \leq 1 - \frac{1}{2n} \\ 0, & \text{if } 1 - \frac{1}{2n} \leq t \leq 1 \end{cases} ; \hat{y}_n(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq 1 - \frac{1}{2n} \\ \text{affine}, & \text{if } 1 - \frac{1}{2n} \leq t \leq 1 \\ 1, & \text{if } t = 1. \end{cases}$$

Clearly $\hat{x}_n, \hat{y}_n \in S(B)$ and $\hat{x}_n \perp \hat{y}_n$. We claim that $\hat{y}_n \in (u + B_0^{**}(u)) \cap \mathcal{B}_B = F_u$. Indeed, the support tripotent of $f(e)$ is bigger than or equal to u in B^{**} , that is, $f(e) = u + P_0(u)(f(e))$ in B^{**} (see (5)). Thus, the support tripotent of \hat{y}_n also is bigger than or equal to u in B^{**} , that is, $\hat{y}_n = u + P_0(u)(\hat{y}_n) \in F_u$.

Therefore, denoting by $x_n = f^{-1}(-\hat{x}_n) \in S(E)$ and $y_n = f^{-1}(\hat{y}_n) \in F_e = (e + E_0^{**}(e)) \cap \mathcal{B}_E = (e + E_0(e)) \cap \mathcal{B}_E$, we have

$$1 = \|\hat{x}_n + \hat{y}_n\| = \|\hat{y}_n - (-\hat{x}_n)\| = \|y_n - x_n\|, \\ 2 - \frac{1}{n} = \|f(e) + \hat{x}_n\| = \|e - x_n\|,$$

and $y_n = e + P_0(e)(y_n)$, for every natural n .

Applying Proposition 2.2 we can find a natural n_0 such that

$$\|P_2(e)(e - x_{n_0})\| > \frac{3}{2} > 1.$$

Finally, the inequalities

$$1 \geq \|P_2(e)(y_{n_0} - x_{n_0})\| = \|P_2(e)(e + P_0(e)(y_{n_0}) - x_{n_0})\| = \|P_2(e)(e - x_{n_0})\| > \frac{3}{2},$$

give the desired contradiction. \square

In our list of ingredients to extend the results in [15], the next goal is a generalization of [15, Lemma 2.4].

Lemma 2.6. *Let E be a JB*-triple. Then every minimal tripotent u in $E^{**} \setminus E$ is orthogonal to all minimal tripotents in E .*

Proof. Suppose, contrary to what we want to prove, that there exist $e \in E$ such that u is not orthogonal to e . The atomic part of E^{**} is precisely the JBW*-subtriple of E^{**} generated by all minimal tripotents in E^{**} (see [16, Theorem 2]) and coincides with an ℓ_∞ -sum of a certain family of Cartan factors (compare [17, Proposition 2]).

We are in position to apply [14, Section 3, page 16, (\checkmark .1) and (\checkmark .2)] to assure that one of the following statements holds:

- (a) There exists a quadrangle (v_1, v_2, v_3, v_4) (that is, $v_1 \perp v_3, v_2 \perp v_4, v_j \in E_1^{**}(v_{j+1}), v_{j+1} \in E_1^{**}(v_j)$ for every $j \in \{1, 2, 3\}$, $v_1 \in E_1^{**}(v_4), v_4 \in E_1^{**}(v_1)$ and $v_4 = 2\{v_1, v_2, v_3\}$) of minimal tripotents in a Cartan factor contained in the atomic part of E^{**} and complex numbers $\alpha, \beta, \gamma, \delta$ such that $e = v_1, |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1, \alpha\delta = \beta\gamma$ and $u = \alpha e + \beta v_2 + \delta v_3 + \gamma v_4$;
- (b) There exists a triangle (e, v, \tilde{e}) (i.e. $e \perp \tilde{e}, e, \tilde{e} \in E_2^{**}(v), v \in E_1^{**}(e), v \in E_1^{**}(\tilde{e})$ and $e = \{v, \tilde{e}, v\}$) with v a rank-2 tripotent and \tilde{e} a minimal tripotent in a Cartan factor contained in the atomic part of E^{**} and complex numbers α, β, δ such that $|\alpha|^2 + 2|\beta|^2 + |\delta|^2 = 1, \alpha\delta = \beta^2$ and $u = \alpha e + \beta v + \delta \tilde{e}$.

By assumptions $e \not\leq u$, and thus $\delta \neq 1$. The minimality of e in E can be combined with Kadison's transitivity theorem (compare [4, Theorems 3.3 and 3.4] and (3)) to deduce that $P_1(e)(E^{**}) \subseteq E$.

In the case (a), we have $v_2, v_4 \in E_1^{**}(e) = P_1(e)(E^{**}) = P_1(e)(E) \subseteq E$ because $e \in E$. Another application of Kadison's transitivity theorem (3) shows that $v_3 \in E_1^{**}(v_2) = P_1(v_2)(E^{**}) = P_1(v_2)(E) \subseteq E$ because $v_2 \in E$. Therefore, v_1, v_2, v_3 and v_4 belong to E and hence $u = \alpha e + \beta v_2 + \delta v_3 + \gamma v_4 \in E$, which is impossible.

In case (b) we have $v \in E_1^{**}(e) = P_1(e)(E^{**}) = P_1(e)(E) \subseteq E$ because $e \in E$. Since v is a rank-2 tripotent, Kadison's transitivity theorem (3) proves that $\tilde{e} \in E_1^{**}(v) = P_1(v)(E^{**}) = P_1(v)(E) \subseteq E$ because $v \in E$. This shows that $u = \alpha e + \beta v + \delta \tilde{e} \in E$ leading us to a contradiction. \square

Our next result is the real core of the study of surjective isometries between the unit spheres of two atomic JBW*-triples.

Theorem 2.7. *Let E and B be atomic JBW*-triples, and suppose that $f : S(E) \rightarrow S(B)$ is a surjective isometry. Then, for each minimal tripotent e in E there exists a unique minimal tripotent u in B such that $f(e) = u$. Moreover, there exists a real linear surjective isometry $T_e : E_0(e) \rightarrow B_0(u)$ such that*

$$f(e + x) = u + T_e(x),$$

for every $x \in \mathcal{B}_{E_0(e)}$, and the restriction of f to the maximal norm-closed face $F_e = e + \mathcal{B}_{E_0(e)}$ is an affine function.

Proof. Combining Proposition 2.3, Theorem 2.4, and (4), we find a minimal tripotent $u \in B^{**}$ such that

$$(6) \quad f(F_e) = f((e + E_0(e)) \cap \mathcal{B}_E) = f((e + E_0^{**}(e)) \cap \mathcal{B}_E) = F_u = (u + B_0^{**}(e)) \cap \mathcal{B}_B.$$

We claim that $u \in B$. If on the contrary $u \in B^{**} \setminus B$, Lemma 2.6 implies that $u \perp v$ for every minimal tripotent $v \in B$. Theorem 2.5 proves that 1 is an isolated point in the triple spectrum of $f(e)$.

As before, we shall identify $B_{f(e)}$ with $C_0(\text{Sp}(f(e)))$. Since 1 is isolated in $\text{Sp}(f(e))$, the element $\hat{w} = \chi_{\{1\}}(f(e))$ is a tripotent in B (actually \hat{w} is the support tripotent of $f(e)$). Having in mind that $f(e) \in F_u$, we have $f(e) = u + P_0(u)(f(e))$ and hence $u \leq \hat{w}$.

Since B is atomic, we can find a minimal tripotent \hat{w}_0 in B satisfying $\hat{w}_0 \leq \hat{w}$. By Lemma 2.6 and the assumptions we have $u \perp \hat{w}_0$, and thus $u \leq \hat{w} - \hat{w}_0$. Clearly,

$$2 = \|f(e) + \hat{w}_0\| = \|f(e) - (-\hat{w}_0)\| = \|e - f^{-1}(-\hat{w}_0)\|$$

and, by Proposition 2.1 [14, Proposition 2.2] or Proposition 2.2, we have

$$w_0 = f^{-1}(-\hat{w}_0) = -e + P_0(e)(w_0).$$

Having in mind that $f(e) - \hat{w}_0 \in (u + B_0^{**}(u)) \cap \mathcal{B}_B$, we deduce that the element $z = f^{-1}(f(e) - \hat{w}_0)$ belongs to $(e + E_0^{**}(e)) \cap \mathcal{B}_E$, and thus $z = e + P_0(e)(z)$, and therefore

$$\begin{aligned} 1 &= \|f(e)\| = \|f(e) - \hat{w}_0 + \hat{w}_0\| = \|(f(e) - \hat{w}_0) - (-\hat{w}_0)\| = \|z - w_0\| \\ &= \|e + P_0(e)(z) - (-e + P_0(e)(w_0))\| = \|2e + P_0(e)(z - w_0)\| = 2, \end{aligned}$$

which is impossible.

We have therefore shown that $u \in B$. We shall now mimic the arguments in the proof of [26, Proposition 3.1], details are enclosed for completeness reasons. Since

$$f(e + \mathcal{B}_{E_0(e)}) = f((e + E_0(e)) \cap \mathcal{B}_E) = F_u = (u + B_0(u)) \cap \mathcal{B}_B = u + \mathcal{B}_{B_0(u)},$$

denoting by \mathcal{T}_{x_0} the translation with respect to x_0 (i.e. $\mathcal{T}_{x_0}(x) = x + x_0$), the mapping $f_e = \mathcal{T}_u^{-1}|_{F_u} \circ f|_{F_e} \circ \mathcal{T}_e|_{\mathcal{B}_{E_0(e)}}$ is a surjective isometry from $\mathcal{B}_{E_0(e)}$ onto $\mathcal{B}_{B_0(u)}$. Mankiewicz's theorem (see [23]) implies the existence of a surjective real linear isometry $T_e : E_0(e) \rightarrow B_0(u)$ such that $f_e = T_e|_{S(E_0(e))}$ and hence

$$f(e + x) = u + T_e(x), \text{ for all } x \text{ in } \mathcal{B}_{E_0(e)}.$$

In particular $f(e) = u$. Now, since

$$f|_{F_e} = \mathcal{T}_u|_{\mathcal{B}_{B_0(u)}} \circ f_e \circ \mathcal{T}_e^{-1}|_{F_e} = \mathcal{T}_u|_{\mathcal{B}_{B_0(u)}} \circ T_e \circ \mathcal{T}_e^{-1}|_{F_e},$$

we deduce that $f|_{F_e}$ is real affine function. \square

We recall now some terminology taken from [3]. Let $K(H, H')$ be the space of all compact linear operators between two complex Hilbert spaces. We shall write $K(H)$ instead of $K(H, H)$. If C_j is a Cartan factor of type $j \in \{1, 2, 3, 4, 5, 6\}$, we define $K_1 = K(H, H')$ for $C_1 = L(H, H')$, $K_j = C_j \cap K(H)$ for $j = 2, 3$, and in the remaining cases $K_4 = C_4$, $K_5 = C_5$, and $K_6 = C_6$. The JB^* -triples K_1, K_2, \dots, K_6 are called elementary JB^* -triples. Suppose $E = \bigoplus_i^\infty C_i$ is an atomic JBW^* -triple, where each C_i is a Cartan factor. It is known that the c_0 -sum $K(E) = \bigoplus_i^{c_0} K_i$ is a weakly compact JB^* -triple and a triple ideal of E with $K(E)^{**} = E$ (see [3, Remark 2.6]). JB^* -triples of the form $K(E)$ are called weakly compact JB^* -triples (see [3]). It is further known that every element x in a weakly compact JB^* -triple $K(E)$ can be written as a norm convergent (possibly finite) sum $x = \sum_{n=1} \lambda_n e_n$, where e_n are mutually orthogonal minimal tripotents in $K(E)$ (and in E), and $(\lambda_n) \subseteq \mathbb{R}_0^+$ with $(\lambda_n) \rightarrow 0$ (see [3, Remark 4.6]).

Now, just a final technical step is separating us from our main goal.

Proposition 2.8. *Let E and B be atomic JBW^* -triples, and suppose that $f : S(E) \rightarrow S(B)$ is a surjective isometry. Then the following statements hold:*

- (a) *For each minimal tripotent $e \in E$ we have $T_e(v) = f(v)$ for every minimal tripotent $v \in E_0(e)$, where $T_e : E_0(e) \rightarrow B_0(f(e))$ is the surjective real linear isometry given by Theorem 2.7;*
- (b) *Let e_1, \dots, e_n be mutually orthogonal minimal tripotents in E , and let $\lambda_1, \dots, \lambda_n$ be positive real numbers with $\max_j \lambda_j = 1$. Then*

$$f\left(\sum_{j=1}^n \lambda_j e_j\right) = \sum_{j=1}^n \lambda_j f(e_j);$$

- (c) *f maps $S(K(E))$ onto $S(K(B))$;*

- (d) For each minimal tripotent $e \in E$ we have $f(u) = T_e(u)$ for every non-zero tripotent $u \in E_0(e)$;
- (e) Let v_1, \dots, v_n be mutually orthogonal non-zero tripotents in E , and let $\lambda_1, \dots, \lambda_n$ be positive real numbers with $\max_j \lambda_j = 1$. Then

$$f\left(\sum_{j=1}^n \lambda_j v_j\right) = \sum_{j=1}^n \lambda_j f(v_j).$$

Proof. (a) Let v be a minimal tripotent in $E_0(e)$. By Theorem 2.7 $f(e)$, $f(-e)$, $f(v)$ and $f(-v)$ are minimal tripotents in B with $\|f(e) - f(-e)\| = \|2e\| = 2$. Lemma 2.1 implies that $f(-e) = -f(e)$. Similarly, $f(-v) = -f(v)$. By hypothesis

$$\|f(e) \pm f(v)\| = \|f(e) - f(\mp v)\| = \|e \pm v\| = 1.$$

By [12, (6) in page 360] we have

$$\text{cp}(\{f(e)\}) = \{y \in B : \|y \pm f(e)\| \leq 1\} = \mathcal{B}_{B_0(e)}.$$

In particular $f(v) \in \mathcal{B}_{B_0(e)}$.

The mapping T_e is a surjective real linear isometry between JBW*-triples. It follows from [20, Theorem 4.8] that T_e preserves the symmetrized triple product $\langle x, y, z \rangle := \frac{1}{3}(\{x, y, z\} + \{z, x, y\} + \{y, z, x\})$. In particular T_e preserves cubes of the form $x^{[3]} = \{x, x, x\}$ and maps minimal tripotents to minimal tripotents. Therefore $T_e(v)$ is a minimal tripotent in $B_0(e)$.

Again by Theorem 2.7 we have

$$f(v) + T_v(e) = f(e + v) = f(e) + T_e(v).$$

Since $f(v) \perp f(e)$, we deduce that

$$f(v) = P_2(f(v))(f(v) + T_v(e)) = P_2(f(v))(f(e) + T_e(v)) = P_2(f(v))(T_e(v)).$$

Since $\|T_e(v)\| = \|v\| = 1$, Lemma 1.6 in [16] implies that $T_e(v) = f(v) + P_0(f(v))T_e(v)$. Finally, since $T_e(v)$ is a minimal tripotent we get $P_0(f(v))T_e(v) = 0$, $T_e(v) = f(v)$.

(b) Let e_1, \dots, e_n be mutually orthogonal minimal tripotents in E , and let $\lambda_1, \dots, \lambda_n$ be positive real numbers with $\max_j \lambda_j = 1$. We may assume $\lambda_1 = 1$. Let T_{e_1} be the surjective real linear isometry given by Theorem 2.7. We deduce from the just quoted Theorem and the statement in (a) that

$$f\left(\sum_{j=1}^n \lambda_j e_j\right) = f(e_1) + T_{e_1}\left(\sum_{j=2}^n \lambda_j e_j\right) = f(e_1) + \sum_{j=2}^n \lambda_j T_{e_1}(e_j) = \sum_{j=1}^n \lambda_j f(e_j).$$

(c) We have already commented that every element in $K(E)$ can be approximated in norm by an element of the form $\sum_{j=1}^n \lambda_j e_j$, where e_1, \dots, e_n are mutually orthogonal minimal tripotents in E , and $\lambda_1, \dots, \lambda_n$ are positive real numbers. Consequently, elements in $S(K(E))$ can be approximated in norm by finite sums of the form $\sum_{j=1}^n \lambda_j e_j$, with e_1, \dots, e_n and $\lambda_1, \dots, \lambda_n$ as above and $\max_j \lambda_j = 1$. If we

observe that, by Theorem 2.7 and (b), $f\left(\sum_{j=1}^n \lambda_j e_j\right) = \sum_{j=1}^n \lambda_j f(e_j) \in S(K(B))$, we

conclude that $f(S(K(E))) \subseteq S(K(B))$. Applying the same argument to f^{-1} we get $f(S(K(E))) = S(K(B))$.

(d) Let u be a non-zero tripotent in $E_0(e)$, where e is a minimal tripotent in E . We may assume that u is not minimal, otherwise the desired statement follows from (a). Thus, since E is atomic, we can find a minimal tripotent v in $E_0(e)$ with $u \geq v$. By Theorem 2.7 we have

$$\begin{aligned} f(e) + T_e(u) &= f(e + u) = f(e + v + (u - v)) = f(v) + T_v(e) + T_v(u - v) \\ &= (\text{by (a)}) = f(v) + f(e) + T_v(u - v) = f(e) + f(v + u - v) = f(e) + f(u), \end{aligned}$$

which proves the desired statement.

(e) Under the assumptions there exists $k \in \{1, \dots, n\}$ with $\lambda_k = 1$. Since E is atomic, we can find a minimal tripotent $e_k \leq v_k$. Let T_{e_k} be the surjective real linear isometry given by Theorem 2.7. We deduce from the just quoted Theorem and the statement in (d) that

$$\begin{aligned} f\left(\sum_{j=1}^n \lambda_j v_j\right) &= f(e_k) + T_{e_k}\left((v_k - e_k) + \sum_{j \neq k} \lambda_j v_j\right) \\ &= f(e_k) + T_{e_k}((v_k - e_k)) + \sum_{j \neq k} \lambda_j T_{e_k}(v_j) \\ &= f(e_k + v_k - e_k) + \sum_{j \neq k} \lambda_j f(v_j) = \sum_{j=1}^n \lambda_j f(v_j). \end{aligned}$$

□

We can establish now our main result.

Theorem 2.9. *Let $f : S(E) \rightarrow S(B)$ be a surjective isometry, where E and B are atomic JBW*-triples. Then there exists a (unique) real linear isometry $T : E \rightarrow B$ such that $f = T|_{S(E)}$.*

Proof. Let $K(E)$ and $K(B)$ denote the ideals of E and B generated by the minimal tripotents in E and B , respectively. We deduce from Proposition 2.8(c) that $f(S(K(E))) = S(K(B))$ and $f|_{S(K(E))} : S(K(E)) \rightarrow S(K(B))$ is a surjective isometry. We observe that $K(E)$ and $K(B)$ are weakly compact JB*-triples in the sense of [3, 14], so by [14, Theorem 2.5] there exists a surjective real linear isometry $S : S(K(E)) \rightarrow S(K(B))$ satisfying $f(x) = S(x)$, for every $x \in S(K(E))$.

The mapping $T = S^{**} : K(E)^{**} = E \rightarrow K(B)^{**} = B$ is a surjective real linear isometry and a weak* continuous mapping. By construction $T(x) = S(x) = f(x)$, for every $x \in S(K(E))$.

We claim that

$$(7) \quad T(w) = f(w), \text{ for every non-zero tripotent } w \in E.$$

Since E is atomic, we can find a minimal tripotent $e \leq w$. The mapping $T_e : E_0(e) \rightarrow B_0(f(e))$ given by Theorem 2.7 is a surjective real linear isometry between (atomic) JBW*-triples. By [24, Proposition 2.3(1.)] T_e also is weak*-continuous. By construction $T_e(v) = f(v) = S(v) = T(v)$ for every finite-rank tripotent $v \in E_0(e)$. Since every tripotent in an atomic JBW*-triple can be approximated in the weak*-topology by a net of finite-rank tripotents, we deduce from the above that $T(w) = T_e(w)$ for every tripotent in $E_0(e)$. Now, by Theorem 2.7 and Proposition 2.8(d) or (e), we have

$$f(w) = f(e) + T_e(w - e) = f(e) + T(w - e) = T(e) + T(w - e) = T(w),$$

which proves the claim.

It is known that in a JBW^* -triple M the set of tripotents in M is norm-total, that is, every element in M can be approximated in norm by finite real linear combinations of mutually orthogonal non-zero tripotents in M (see [19, Lemma 3.11]. However, a general JBW^* -triple need not contain a single minimal tripotent). Combining this fact with (7) and Proposition 2.8(e), we can easily conclude that $T(x) = f(x)$, for every $x \in S(E)$. \square

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